For both we use the second-order prefilter $F(s) = 2500/(s^2 + 70s + 2500)$ and a first-order Padé approximation for the zero-order hold $H(s)$. The sampling period is $T = \pi/100$ and the cutoff frequency $\omega_c = 6.9$ rad/s.

Using the approach in Sec. III, the optimum errors ϵ in Eq. (7) are in the order of 10^{-3} with the frequency range partitioned in 200 points. The digital controllers obtained had order five. Plots comparing the analog and sampled-data, closed-loop step responses are shown in Figs. 2 and 3, which present an improvement over all the methods mentioned in Sec. I.

For the general approach, the first example was considered using different starting points. It was observed that there is no significant reduction of the cost with respect to the first method, the initial being $\epsilon_0 = 0.0043$, which after ten iterations was reduced to $\epsilon = 0.0029$.

Acknowledgments

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State-Variable Models of Structures Having Rigid-Body Modes

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Introduction

THE equation of motion of structures is usually written in the form

$$M\ddot{x} + C\dot{x} + Kx = f_x \tag{1}$$

where M is the mass matrix, C is the damping matrix, and K is the stiffness matrix, all of order $(n \times n)$; and where x is the displacement vector and f_x the force vector, both of order $(n \times 1)$.[1] It will be assumed that M is positive definite, although the results could be generalized to the case of positive semidefinite M. If the structure has rigid-body freedom, K is singular. Equation (1) may be expanded to $2n$-order state-space form as follows:

$$A\dot{z} + Bz = f_z \tag{2}$$

where

$$A \equiv \begin{bmatrix} 0 & M \\ M & C \end{bmatrix}, \quad B \equiv \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}, \quad z \equiv \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix}, \quad f_z \equiv \begin{Bmatrix} 0 \\ f_x \end{Bmatrix} \tag{3}$$

Let (λ_i, ϕ_i) be an (eigenvalue, right-eigenvector)-pair associated with the eigenproblem

$$(\lambda_i A + B)\phi_i = 0, \qquad i = 1,2,\ldots,2n \tag{4}$$

This paper, which can be considered a supplement to Ref. 2, discusses some interesting features of the state-variable, rigid-body modes that arise when B is singular.

Generalized Eigenvectors, Jordan Form

An $n \times n$ matrix A that fails to have a linearly independent set of n eigenvectors is said to be defective. This may occur when A has a repeated eigenvalue. It is then not possible to transform A into diagonal form; i.e., there exists no Φ such that

$$A\Phi = \Phi\Lambda \tag{5}$$

where Λ is a diagonal matrix. But it is possible to find a linearly independent set of generalized eigenvectors, which transform A into the almost-diagonal Jordan form

$$AQ = QJ \tag{6}$$

Reference 3 defines these concepts and shows, for example, that when A has a repeated eigenvalue λ_2 of multiplicity three and an eigenvalue λ_3 of multiplicity two, the Jordan matrix will have the form

$$J = \left[\begin{array}{c|ccc|cc|c} \lambda_1 & & & & & & \\ \hline & \lambda_2 & 1 & 0 & & & \\ & 0 & \lambda_2 & 1 & & & \\ & 0 & 0 & \lambda_2 & & & \\ \hline & & & & \lambda_3 & 1 & \\ & & & & 0 & \lambda_3 & \\ \hline & & & & & & \lambda_4 \end{array}\right] \tag{7}$$

where the repeated eigenvalues lead to Jordan blocks having the eigenvalue on the diagonal and ones on the superdiagonal.

It will now be shown that systems which have rigid-body modes and which are described by the state-variable equation of the form of Eq. (2) require the use of generalized eigenvectors.

Undamped Systems with Rigid-Body Modes

Let an undamped system be described by the physical equation of motion

$$M\ddot{x} + Kx = 0 \tag{8}$$

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and let this system have n_r rigid-body modes; i.e., K is of rank $(n-n_r)$. Then, the n_r rigid-body displacement modes can be obtained from Eq. (1):

$$K\Phi_r^D = \mathbf{0}_{nr} \tag{9}$$

where Φ_r^D is an $(n \times n_r)$ matrix of rigid-body displacement modes.

Now, let the same undamped system be described by the state-variable equation

$$A\dot{z} + Bz = \mathbf{0} \tag{10}$$

where, from Eqs. (3),

$$A = \begin{bmatrix} 0 & M \\ M & C \end{bmatrix}, \qquad B = \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \tag{11}$$

The corresponding eigenproblem is

$$\lambda A\phi + B\phi = 0 \tag{12}$$

and the characteristic equation for λ is

$$\det(\lambda A + B) = 0 \tag{13}$$

If state rigid-body modes are defined as those state vectors that satisfy Eq. (12) with $\lambda = 0$, then these state rigid-body modes must satisfy the equation

$$B\phi \equiv \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \begin{Bmatrix} \phi^V \\ \phi^D \end{Bmatrix} = \begin{Bmatrix} -M\phi^V \\ K\phi^D \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{14}$$

where superscripts D and V refer to the displacement and velocity partitions of the state vector z, respectively. If M is nonsingular, then ϕ^V must be zero, and the only state rigid-body modes will have the form

$$\phi_r' = \begin{Bmatrix} \mathbf{0} \\ \phi^D \end{Bmatrix} \tag{15}$$

and there will only be n_r such modes. On the other hand, the eigenvalue $\lambda = 0$ will occur as a root of multiplicity $2n_r$ of Eq. (13). Thus, the generalized eigenproblem of Eq. (12) is defective. It possesses one set of n_r *regular state* rigid-body modes given by

$$\Phi_r' = \begin{bmatrix} 0_{nr} \\ \Phi_r^D \end{bmatrix}_{2n \times n_r} \tag{16}$$

where Φ_r^D is given by Eq. (9). For each column of Φ_r', i.e., each regular state rigid-body mode corresponding to $\lambda = 0$, there will be a *generalized state* rigid-body mode, ϕ_r'', defined by a generalization of Eq. (6), namely

$$A[\phi_r' \phi_r''] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + B[\phi_r' \phi_r''] = [\mathbf{0} \quad \mathbf{0}] \tag{17}$$

or

$$B\phi_r' = \mathbf{0} \tag{18}$$

$$B\phi_r'' = -A\phi_r' \tag{19}$$

Equation (18) defines the regular state rigid-body modes that individually have the form of Eq. (15) and that collectively form Φ_r', whereas Eq. (19) defines the corresponding generalized state rigid-body modes. The set of n_r generalized state rigid-body modes Φ_r'' corresponding to the regular state rigid-body modes Φ_r' is found by expanding Eq. (17) to obtain

$$A[\Phi_r' \Phi_r''] \begin{bmatrix} 0_{rr} & I_{rr} \\ 0_{rr} & 0_{rr} \end{bmatrix} + B[\Phi_r' \Phi_r''] = [0_{sr} \quad 0_{sr}] \tag{20}$$

where subscript s stands for the $2n$ rows of state vectors. Hence,

$$B\Phi_r' = 0_{sr} \tag{21}$$

$$B\Phi_r'' = -A\Phi_r' \tag{22}$$

Equations (11), (16), and (22) may be combined to obtain

$$\begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} \Phi_r''^V \\ \Phi_r''^D \end{bmatrix} = -\begin{bmatrix} 0 & M \\ M & 0 \end{bmatrix} \begin{bmatrix} 0_{nr} \\ \Phi_r^D \end{bmatrix} \tag{23}$$

The row partitions of Eq. (23) are

$$-M(\Phi_r''^V - \Phi_r^D) = 0_{nr} \tag{24}$$

$$K\Phi_r''^D = 0_{nr} \tag{25}$$

When M is nonsingular, Eq. (24) requires that $\Phi_r''^V = \Phi_r^D$. The $\Phi_r''^D$ satisfies the same equation as Φ_r^D; i.e., Eq. (25) is the same as Eq. (9). However, $\Phi_r''^D$ can also be set equal to zero, in which case Φ_r'' is given by

$$\Phi_r'' = \begin{bmatrix} \Phi_r^D \\ 0_{nr} \end{bmatrix} \tag{26}$$

Thus, the complete set of $2n_r$ state rigid-body modes is given by

$$\Phi_r \equiv [\Phi_r' \Phi_r''] = \begin{bmatrix} 0_{nr} & \Phi_r^D \\ \Phi_r^D & 0_{nr} \end{bmatrix} \tag{27}$$

where Φ_r^D is given by Eq. (9).

Damped Lumped-Mass Systems with Rigid-Body Modes

Assume that there is a damped system described by Eq. (1), and assume that its stiffness matrix is of rank $(n-n_r)$. Then, the eigenproblem

$$(\lambda^2 M + \lambda C + K)\phi^D = \mathbf{0} \tag{28}$$

has n_r eigenvalues $\lambda = 0$ and n_r corresponding rigid-body displacement modes given by Eq. (9). Thus, as in the undamped case, there will be n_r regular state rigid-body modes given by Eq. (16). Equations (22) again defines the corresponding generalized state rigid-body modes. However, in the damped case, A has the form given in Eq. (3) so that, assuming that Φ_r'' is not identically zero, Eq. (22) becomes

$$\begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} \Phi_r''^V \\ \Phi_r''^D \end{bmatrix} = -\begin{bmatrix} 0 & M \\ M & C \end{bmatrix} \begin{bmatrix} 0_{nr} \\ \Phi_r^D \end{bmatrix} \tag{29}$$

Thus,

$$-M(\Phi_r''^V - \Phi_r^D) = 0 \tag{30}$$

$$K\Phi_r''^D = -C\Phi_r^D \tag{31}$$

Since M is assumed to be nonsingular, $\Phi_r''^V = \Phi_r^D$ as in the undamped case. Since K is singular, Eq. (31) will have a solution only for certain forms of C. For example, if any of the columns of $C\Phi_r^D$ is zero, there will be a solution for the corresponding column of $K\Phi_r''^D$, just as for the undamped case. However, for those columns of $C\Phi_r^D$ which are not zero, there will not be a solution for corresponding columns of $K\Phi_r''^D$, and hence the assumption that $\lambda=0$ is a double root leading to both regular and generalized state rigid-body modes is not valid in this case.

Examples

Undamped Lumped-Mass Systems

As an example of the preceding theory of state rigid-body modes, consider the undamped two-mass system of Fig. 1.

Then,

$$A=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix},\quad B=\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&-1\\0&0&-1&1\end{bmatrix}$$

The eigenvalues are $\lambda=0,\ 0,\ j\sqrt{2}, -j\sqrt{2}$. Since the rank of $(\lambda_1 A+B)$ for $\lambda_1=0$ is 3, the null space of $(\lambda_1 A+B)$ has dimension 1. Therefore, there is only one regular eigenvector corresponding to $\lambda_1=0$. This is given by $B\phi_1=\mathbf{0}$; therefore

$$\phi_1^T=[0\ 0\ 1\ 1]$$

From Eq. (22)

$$B\phi_2=-A\phi_1$$

which gives

$$\phi_2^T=[1\ 1\ 0\ 0]$$

The full set of linearly independent eigenvectors (modes) for this example problem is given by the modal matrix

$$\Phi=\begin{bmatrix}0&1&j\sqrt{2}&-j\sqrt{2}\\0&1&-j\sqrt{2}&j\sqrt{2}\\1&0&1&1\\1&0&-1&-1\end{bmatrix}$$

and the corresponding Jordan matrix is

$$\Lambda=\left[\begin{array}{cc:cc}0&1&0&0\\0&0&0&0\\ \hdashline 0&0&j\sqrt{2}&0\\0&0&0&-j\sqrt{2}\end{array}\right]$$

(Note the Jordan block corresponding to the repeated root $\lambda=0$.)

A physical significance can be attributed to the first two columns of Φ; namely, rigid-body motion with arbitrary velocity can be expressed as a linear superposition of these two columns.

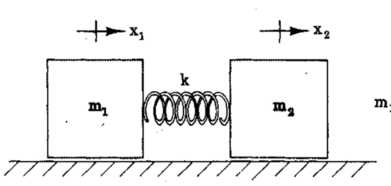

Fig. 1 Two-mass undamped system.

Damped Lumped-Mass Systems

Consider the damped three degree-of-freedom (3 DOF) system shown in Fig. 2.

Case I: A Case for Which $C\phi_r^D=0$

Let $c_1=0.2$, $c_2=0.4$, and $c_3=0$. Then,

$$M=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad K=\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}$$

$$C=\begin{bmatrix}0.2&-0.2&0.0\\-0.2&0.6&-0.4\\0.0&-0.4&0.4\end{bmatrix}$$

The characteristic equation for this system is

$$\lambda^2(\lambda^4+1.2\lambda^3+4.24\lambda^2+1.8\lambda+3)=0$$

so $\lambda=0$ is a root of multiplicity 2. The rigid-body displacement mode ϕ_1^D based on Eq. (9) is

$$\phi_1^D=[1\ 1\ 1]^T$$

Even though C is not proportional to K, it is seen that $C\phi_1^D=\mathbf{0}$, which is consistent with the fact that $\lambda=0$ has multiplicity 2. Consequently, this system will have a state rigid-body mode matrix given by Eq. (27).

Case II: A Case for Which $C\phi_r^D\neq 0$

Let $c_1=0.2$, $c_2=0.4$, and $c_3=0.2$. M and K remain the same as in case I, and

$$C=\begin{bmatrix}0.2&-0.2&0.0\\-0.2&0.6&-0.4\\0.0&-0.4&0.6\end{bmatrix}$$

For this case the characteristic equation is

$$\lambda(\lambda^5+1.4\lambda^4+4.4\lambda^3+2.416\lambda^2+3.12\lambda+0.2)=0$$

so $\lambda=0$ is a single root. The rigid-body displacement mode is the same as in case I, but in the present case, $C\phi_1^D\neq 0$; therefore the system is nondefective, and there is no generalized eigenvector corresponding to $\lambda=0$.

Finite Element Model with Nonclassical Damping

Reference 2 describes a component mode synthesis method for model-order reduction of nonclassically damped systems. In that paper, state rigid-body modes were required for the finite element model of a free-free uniform beam with assumed nonclassical damping. Figure 3 shows such a structural model.

Although the damping matrix of each element is assumed to be proportional to the element's stiffness matrix, the system as

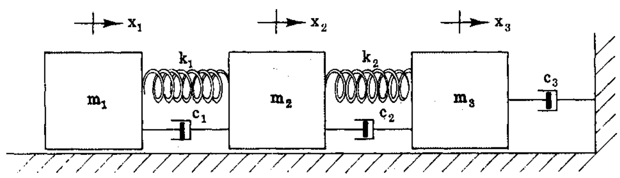

Fig. 2 Three-degree-of-freedom viscous-damped system.

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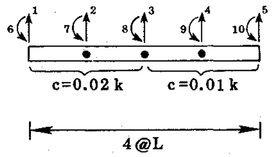

Fig. 3 Free-free beam with nonclassical damping.

a whole has nonproportional damping. The system stiffness matrix K is singular and $n_r = 2$. For each rigid-body mode, $C\phi_r^D = \mathbf{0}$. Therefore, this system requires both regular and generalized state rigid-body modes, so the state rigid-body modes are given by Eq. (27), where

$$\Phi_r^D = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & L & 2L & 3L & 4L & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

Summary

When the equations of motion of a structure having rigid-body freedom are cast in state-variable form, generalized state rigid-body modes may be required. The equations governing these generalized eigenvectors have been given, and examples of undamped and damped structures have been presented.

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Stability Condition for Flexible Structure Control with Mode Residualization

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Introduction

For the structural control or flutter suppression problem, the dynamic model of a structurally flexible system necessarily includes a large number of modes (theoretically infinite). However, only a limited number of modes can be accommodated in controller design. One of the methods for obtaining a reduced-order controller is to use the formalism of mode residualization. This is extensively applied in the flutter suppression[1] and structural control[2,3] problems. This formulation of the reduced-order model is obtained by assuming that the residual modal states of the system quickly reach steady state values. The reduced-order controller design is then based on this simplified model. Although such a controller usually works well on the reduced-order system, it may not stabilize the actual closed-loop system. This is, of course, the most important evaluation criterion for a control law selection.

In this Note, we develop the general idea of mode residualization for a high-order, flexible structure and obtain results for the operation of an observer-based controller, based on the reduced-order model, in closed-loop with the actual system. By applying the perturbation theory, we derive a new stability condition on the basis of the frequency domain representation, which guarantees the stability of the overall closed-loop system even when the sensors and actuators are not collocated. In the flutter suppression or other structural control problems, the natural frequency, mode shape, and damping characteristics of modes are all known imprecisely. Thus, parameter uncertainty may cause the system to be unstable. Addressing this issue, we also present a simple robust stability condition, which insures the overall stability so that the controlled system will not be destabilized by parametric perturbations.

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System Description

Consider the partitioned state-space representation of linear time-invariant systems having the following form

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \tag{1a}$$

$$\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) \tag{1b}$$

$$y(t) = C_1x_1(t) + C_2x_2(t) \tag{1c}$$

in which $x_1(t) \in R^{n_1}$ and $x_2(t) \in R^{n_2}$ represent the model state vector and residual mode state vector, respectively; $u(t) \in R^m$ is the control vector; $y(t) \in R^r$ is the output vector; A_{11}, A_{12}, A_{21} and A_{22} are, respectively, $n_1 \times n_1$, $n_1 \times n_2$, $n_2 \times n_1$ and $n_2 \times n_2$ plant matrices; B_1 and B_2 are, respectively, $n_1 \times m$ and $n_2 \times m$ input matrices; C_1 and C_2 are, respectively, $r \times n_1$ and $r \times n_2$ output matrices.

If A_{22} is nonsingular, by assuming small changes in the residual modal states $x_2(t)$ (i.e., $\dot{x}_2(t) \to 0$), we have the following reduced-order system

$$\dot{\bar{x}}_1(t) = A\bar{x}_1(t) + Bu(t) \tag{2a}$$

$$\bar{y}(t) = C\bar{x}_1(t) + Du(t) \tag{2b}$$

where

$$A = A_{11} - A_{12}A_{22}^{-1}A_{21}, \qquad B = B_1 - A_{12}A_{22}^{-1}B_2$$

$$C = C_1 - C_2A_{22}^{-1}A_{21}, \qquad D = -C_2A_{22}^{-1}B_2$$

Note that although the full-order system [Eq. (1)] does not have a direct-feed term [D matrix in Eq. (2b)], the reduced-order system [Eq. (2)] may have the outputs directly affected by the inputs [i. e., the term $Du(t)$]. In general, this coefficient is small and may result in only a slight change in the low-frequency behavior of the mode.[1] Without loss of generality, assume that the reduced-order system (2) is controllable and observable. The observer-based controller based on Eq. (2) is assumed to be of the form

$$u(t) = -G\hat{x}_1(t) \tag{3a}$$